The Schlesinger system and isomonodromic deformations of bundles with connections on Riemann surfaces

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Abstract

We introduce a way of presentation of pairs (E, ∇), where E is a bundle on a Riemann surface and ∇ is a logarithmic connection in E, which is based on a presentation of the surface as a factor of the exterior of the unit disc. In this presentation we write the local equation of isomonodormic deformation of pairs (E, ∇). These conditions are written as a modified Schlesinger system on a Riemann sphere (and in the typical case just as an ordinary Schlesinger system) plus some linear system.

1 Introduction

Let us be given a Fuchsian system of differential equations on a Riemann sphere:

$$\frac{dy}{dz} = \sum_i \frac{B_i}{z - a_i} y.$$

Let us change locations of singularities a_i in such a way that the monodromy is preserved and the singularities do not confluence. Then the residues B_i become multivalued functions of a_i, and the condition that the family of systems

$$\frac{dy}{dz} = \sum_i \frac{B_i(a_1, \ldots, a_n)}{z - a_i} y$$

is isomonodromic can be written as a system of nonlinear differential equations for the functions B_i. In particular if one restricts to the Schlesinger deformations (mention that in typical case all deformations are Schlesinger, see [1]), than this system is the Schlesinger system.

Take instead of the Riemann sphere a Riemann surface of positive genus. In this case it is natural to consider not the deformations of linear systems (in other terms, connections in a trivial bundle), but isomonodromic deformations

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of bundles with connections. Both a connections and a bundle are allowed to change. The motivation for this point of view can be found in the introduction to the paper [2]. It is also natural to allow to change a module of a complex structure. Such deformations were considered by different authors in [3]–[9] and also in many other papers.

As usual special cases are considered, for example surfaces of genus 1. Krichever wrote equations that describe deformations in the general case [5]. His approach is based on meromorphic triviality of bundles on Riemann surfaces, however equations that are obtained in [5], differ much from the Schlesinger system.

In the case of genus 1 another approach to the description of isomonodromic deformations is known: the elliptic Schlesinger system (see for example papers [2], [10]), which is some system of equation describing isomonodromic deformation of bundles with connections on a torus; this system is a generalization of the Sclesinger system . In the paper [2] the author says that is desirable to generalize this construction to the case of higher genus and to write in this case an analog of the Schlesinger system. We fins this this generalization in the present paper. In particular we prove that in the case of Riemann surfaces isomonodromic deformations can be described by the Schlesinger system plus some system of linear equation.

The paper is organized as follows. In section 2 the space of parameters of derormations is described: this the Teichmuller space with marked points that are locations of singularities. We take the Teichmuller space not the space of modules by the following reason: in order to speak about monodromy matrix corresponding to bypasses along canonical cuts, one must fix somehow canonical cuts, but the pairs "a complex structure+a system of canonical cuts" modulo some equivalence form the Teichmuller space. For an explicit description of the space of parameters of the deformation we present the Riemann surface as factor of the exterior of a unit disk by the action of a Fuchsian group. Then we choose in a canonical way the fundamental polygon with $4g$ vertices (g is the genus of the surface). The singularities correspond to some points of the polygon.

In the section 3 we suggest a way of description of bundles with connections on a Riemann surface. They are described by the following data: a form ω on a Riemann sphere and a collection of nondegenerate matrices $S_{x_0^1,x_0^i}$, $i = 2,\dots,4g$, where indices x_0^i correspond to the vertices of the fundamental polygon. The form ω is constructed as follows. There exists a factorizing map from the fundamental to the Riemann surface. Take an inverse of the bundle with connection on the Riemann surface. The obtained bundle with connection on the fundamental polygon is continued to a bundle with a connection on the whole Riemann sphere with an additional singularity in the zero. We fix then a meromorphic trivialization of this bundle on the Riemann sphere, which is holomorphic on $\mathbb{C}\setminus\{0\}$. In this trivialization the connection is defined by a form, this is the form ω (see. sec. 3.1).

Explain the construction of the matrices $S_{x_0^1,x_0^i}$. The bundle with a connection on the fundamental polygon is an inverse image of the bundle with the

connection on the Riemann surface. All vertices of the polygon are glued togather. Hence there exists an operator that identifies stalks over vertices x_0^1 and x_0^i. The matrix $S_{x_0^1,x_0^i}$ is the matrix of this operator (see sec. 3.2).

The constructed data is sufficient for a reconstruct a bundle with a connection on the surface (see theorem 1). Note that the procedure of construction of the form and matrices is non-canonical: different forms and matrices can give equivalent bundles. But it is well-known that all possible bundles do not form any "good" space (see discussion in the paper [11]), that is why there is no way of description of bundles with connections. Essentially we consider not the deformations of bundles with connections, but the deformations of data, introduced above (note that in the paper [5] instead of bundles with connections actually the parameters of meromorphic trivialization named Turin parameters are considered instead of).

In the section 4 the Schlesinger isomonodromic deformations of bundles with connections on a Riemann surface are defined and the equations are obtained, that describe evolution of data, introduced in section 3, under the Schlesinger isomonodromic deformations. It is proved that the isomonodromic deformations are described by a system of nonlinear equations for the coefficients of the form ω (in typical case this just the Schlesinger system) and some linear system for the matrices $S_{x_0^1 x_0^i}$ (see. Theorem 2). The Sclesinger system can be presented as a Hamiltonian system. Thus the approach suggested in the present paper does not lead to the appearance of new integrable systems.

The relations between the present approach and Krichever's approach from [5] are considered in the section 5.

2 The space of parameters of defomations and the deformed objects

2.1 The space of parameters

Let M be a Riemann surface of genus $g > 1$, the case of genus $g = 0$, $g = 1$ we consider trivial. The aim of the present paper is to find an analog of the Schlesinger system (ordinary or elliptic), well-known for genuses $g = 0$, $g = 1$, in the case of higher genus.

Fix an initial point x_0 on M.

Definition 1. Let T be the Teichmuller space with n marked points $a_1, \dots, a_n$, where $a_i \neq a_j$ for $i \neq j$. The space $\widetilde{T}$ of parameters of deformations is the universal covering of the space T.

Take an image of a point $\tau \in \widetilde{T}$ in the space T. Then one can speak about marked points, corresponding to τ, and also about a complex structure and a system of canonical cuts corresponding to the point τ, in other words, about a point of the Teichmulle space (without marked points), corresponding to the

point τ. Below we shall call the Teichmuller space the ordinary Teichmuller space without marked points, its points we shall call marked Riemann surfaces.

The space $\widetilde{T}_1$, on which isomonodromic families of pairs "a bundle+a connection", is constructed as follows.

Definition 2. Let T_1 be the Teichmuller space with $n+1$ marked points z, $a_1, \ldots, a_n$, where $a_i \neq a_j$ for $i \neq j$. The space $\widetilde{T}_1$ is the univeral covering of the space T_1 by the variables a_i.

There exists a mapping $\widetilde{T}_1 \to \widetilde{T}$, "forgetting" the marked point z.

Definition 3. Let $\tau \in \widetilde{T}$. Denote by $\widetilde{T}_1|_\tau$ the preimage of the point τ under the mapping $\widetilde{T}_1 \to \widetilde{T}$.

The space $\widetilde{T}_1|_\tau$ can be viewed as a Riemann surface with a complex structure, canonical cuts and marked points $a_1, \ldots, a_n$, defined by the point $\tau \in \widetilde{T}$.

The universal covering is taken to provide the global existence of Schlesinger deformations for every initial condition (see the proposition 5 in section 4). In the present paper we write only local equations of isomonodromic deformations. If one considers only small changes of parameters one can use T instead of $\widetilde{T}$ as the space of parameters, then parameters are just locations of singularities and a point in the Teichmuller space. The deformed objects are pairs "a bundle+a connection", not a form (a system of linear equations or, equivalently, a connection in a trivial bundle as in the case of genus 0). When we change positions of singularities both a bundle and a connection are changing. The evolution of a bundle and a connection is defined by the change of a point in the Teichmuller space and of locations of singularities.

2.2 The fundamental polygon

Let D be the exterior of the unit disc. A point in the Teichmuller can be defined by a collection of automorphisms $Q_1, \ldots, Q_{2g} \in \operatorname{Aut} D$, that satisfy the following properties [12]:

1) the equality $\prod_j Q_{2j} Q_{2j-1} Q_{2j}^{-1} Q_{2j-1}^{-1} = 1$ holds;

2) the subgroup $G \subset \operatorname{Aut} D$, generated by $Q_1, \ldots, Q_{2g}$, is Fuchsian;

3) there exists a fundamental domain of the action of the subgroup $G \subset \operatorname{Aut} D$ on D, which does not intersect ∂D.

The surface M is reconstructed as a factor of the exterior of the unit disc under the action of the subgroup $G \subset \operatorname{Aut} D$, generated by automorphisms $Q_1, \ldots, Q_{2g}$. This action has a fundamental domain, which is non-euclidean polygon U with $4g$ edges ("noneucledean" means that every edge is a noneucledean line). Canonical cuts are reconstructed as images of edges of the fundamental polygon.

Two collections $Q_1, \ldots, Q_{2g}$ and $K_1, \ldots, K_{2g}$ define the same point in the Teichmuller space if and only if there exists an automorphism Q such that $K_1 = QQ_1Q^{-1}, \ldots, K_{2g} = QQ_{2g}Q^{-1}$. There exist a normalized way to choose

a collection of automorphisms $Q_1,\dots,Q_{2g}$, such that it is reconstructed canonically from a point in the Teichmuller space. A traditional way of normalization is described in [12]. When a normalization is fixed one can construct a canonical fundamental polygon. Its vertices x_0^i depend smoothly (but non complex-analytic) on a point in the Teichmuller space.

Change a traditional way of normalization in such a way that $x_0^1 \equiv \infty$ become independent from the point in the Teichmuller space. In order to do it let us change canonically an automorphism Q_z - of the set D, that maps z to $\infty \in D$. Define $Q_1,\dots,Q_{2g}$ as a normalized collection of generators in a traditional sense, and let x_0^1 be a vertex of the fundamental polygon U. Take new generators $Q_{x_0^1}Q_1Q_{x_0^1}^{-1},\dots,Q_{x_0^1}Q_{2g}Q_{x_0^1}^{-1}$. They define a point in the Teichmuller space. Also $Q_{x_0^1}(U)$ is a fundamental polygon for the action of the Fuchsian group given by a new set of generators. The first vertex of the polygon $Q_{x_0^1}(U)$ is ∞.

We have proved.

Proposition 1. *There exists a canonical way of choosing a fundamental polygon such that it (i.e. coordinates of its vertices) depend smoothly (not complex-analytic) on a point in the Teichmuller space, and one of its vertices is always ∞.*

This polygon is denoted below as U, and its vertices as x_0^i, $i = 1,\dots,4g$, one has $x_0^1 = \infty$. The marked points $(z, a_1,\dots,a_n)$ become points in U.

When we are studying deformations in the section 4 we do not allow the singularities to cross the canonical cuts, this is not essential since we are considering only local deformations.

3 Description of bundles with connections on a Riemann surface

Let E be a bundle on a surface M and ∇ a connection in E with singularities in $a_1,\dots,a_n \in M$. Suppose that an initial point $x_0 \in M$ is nonsingular. In the present section for a pair (E,∇) on a Riemann surface we construct a form on a Riemann sphere and some matrices. From them one can reconstruct a pair (E,∇) on a surface. The construction of the form is noncanonical since at some moment we fix a trivialization of some bundle.

3.1 The construction of the form on a Riemann sphere

Construct a form on a fundamental polygon. We have presented a marked Riemann surface as a factor of the fundamental polygon U. Let (E_U,∇_U) be an inverse image on U of the pair (E,∇) under the factorization. Continue the pair (E_U,∇_U) until the pair $(E_{\overline{\mathbb{C}}},\nabla_{\overline{\mathbb{C}}})$ on the whole Riemann sphere. To do it let us calculate a monodromy of the connection ∇_U corresponding to the bypass along $\gamma = \partial U$. One can easily see that the monodromy equals $M_\gamma = M_{a_1}\dots M_{a_n}$. We

denote the monodromy of ∇ corresponding to the bypass along γ as M_γ, and the monodromy corresponding to the bypass around a_i as M_{a_i}, $i = 1, \dots, n$.

Take in the domain $\overline{\mathbb{C}} \setminus U$ a trivial bundle E'. Take as ∇' a connection with the only singularity and the monodromy M_γ of the bypass around zero. Thus on the boundary ∂U the bundles E_U and E' are trivial and connections ∇_U and ∇' in them have the same monodromy. From here we conclude that we can glue pairs (E_U, ∇_U) and (E', ∇') into a pair $(E_{\overline{\mathbb{C}}}, \nabla_{\overline{\mathbb{C}}})$ on the whole Riemann sphere. It can be obtained by gluing them over horizontal sections over ∂U. Let us describe this procedure of gluing since we shall use it several times.

Proposition 2. *Let V be a domain on a Riemann surface and γ be a nonclosed curve without self intersections that cuts the domain into two parts V' and V''. Let us be given two pairs: (E', ∇') on V' and (E'', ∇'') on V'' without singularities on γ. Fix an identifications $E'_P = E''_P$ over some point $P \in \gamma$. Then there exists a uniquely defined procedure of gluing of pairs (E', ∇') and (E'', ∇'') into a pair (E, ∇) on V. If a curve is closed, then the gluing is possible if and only if the monodromies of ∇' and ∇'' along γ are the same.*

of the proposition is well-know, we omit it. □

Every bundle on a Riemann sphere is meromorphically trivial. Moreover there exists a meromorphic trivialization, which is holomorphic on $\overline{\mathbb{C}} \setminus \{0\}$. Fix such a trivialization. The sections become holomorphic vector-columns. The connection $\nabla_{\mathbb{C}}$ can be defined using a form

$$\omega = \left(\frac{C_k}{z^k} + \cdots + \frac{C_1}{z} + \sum_i \frac{B_i}{z - a_i} \right) dz \tag{1}$$

with a regular singularity in zero.

Definition 4. Let (E_U, ∇_U) be the inverse image of the pair (E, ∇) under the factorization $U \to M$. Then ω is the form of the connection ∇_U in such a trivialization of E_U. It is of the type (1) with a regular singularity in zero.

Note that such a trivialization is not unique.

Remark 5. For a typical monodromy matrices and typical positions of singularities one can take such a trivialization that the form is written as

$$\omega = \left(\frac{C_1}{z} + \sum_{i=1}^{n} \frac{B_i}{z - a_i} \right) dz. \tag{2}$$

In this situation $C_1 = -\sum_{i=1}^n B_i$. If we put $a_0 = 0$ and $B_0 = -\sum_{i=1}^n B_i$, then

$$\omega = \sum_{i=0}^{n} \frac{B_i}{z - a_i} \, dz.$$

3.2 The construction of the matrices of gluing operators $S_{x_0^1 x_0^i}$

Introduce additional objects. Using them and the form ω one can reconstruct a pair (E, ∇) on a Riemann surface. To reconstruct a bundle on a Riemann surface we need operators $S_{z,z'} \colon E_{U,z} \mapsto E_{U,z'}$ from a stalk of the bundle E_U over a point z to the stalk of E_U over a point z'. These operators are defined in the following way for every ordered pair of points $z, z' \in \partial U$, that are glued under the factorization $U \to M$.

Definition 6. The bundle E_U is an inverse image under the factorization of the bundle E on M, hence there exist isomorphisms of stalks $E_{Uz} \to E_Z$ and $E_{Uz'} \to E_Z$. Define $S_{z,z'}$ as $E_{U,z} \to E_Z \to E_{U,z'}$, where the second mapping is the inverse to $E_{U,z'} \to E_Z$.

But it is excess to know all operators $S_{z,z'}$. Let $x_0^1, \dots, x_0^{4g}$ be the vertices of the fundamental polygon. Below we show that it is sufficient to know only the operators $S_{x_0^1, x_0^i}$. Since the trivialization of the bundle E_U is fixed, we speak below about the matrices $S_{x_0^1, x_0^i}$.

Definition 7. The matrices $S_{x_0^i, x_0^j}$ are matrices of operators, that glue the stalks over points x_0^i, x_0^j in the sense of the definition 6.

Thus, for the bundle with a connection on a Riemann surface M and an initial point x_0 we have constructed ω of type (1) with singularities $a_i \in U$, $i = 1, \dots, n$, and a regular singularity at zero, and matrices $S_{x_0^1, x_0^i}$, $i = 1, \dots, 4g$.

3.3 The reconstruction of a bundle with a connection from the form ω and matrices $S_{x_0^1, x_0^i}$

At first we suppose that we are given a form and matrices that are obtained from a pair on a Riemann surface and consider the procedure of the reconstruction of a pair on a Riemann surface. Then we investigate the question when such a reconstruction is possible.

At first we construct a pair (E_U, ∇_U) on the fundamental polygon: a bundle E_U is a trivial bundle $U \times \mathbb{C}^p$, and ∇_U is a connection in it, defined by the form ω. Reconstruct operators $S_{z,z'}$ for every pair of points $z, z' \in \partial U$ that are glued under the factorization in the surface. The points $z, z' \in \partial U$ belong to edges $x_0^i x_0^{i+1}$ and $x_0^{j+1} x_0^j$ that are glued (the order means that the edges with the opposite factorization).

Proposition 3. *Let Y_1 be a matrix, whose columns are horizontal sections of the bundle E_U over the edge $x_0^i x_0^{i+1}$ with the initial condition $Y_1(x_0^i) = E$. Let Y_2 -be a matrix, whose columns are horizontal sections of E_U over the edge $x_0^{j+1} x_0^j$ with the initial condition $Y_2(x_0^{j+1}) = S_{x_0^i x_0^{j+1}} = S^{-1}_{x_0^1 x_0^i} S_{x_0^1 x_0^{j+1}}$. Then $S_{z,z'} = Y_2(z') Y_1(z)^{-1}$.*

Proof. Since the edges $x_0^i x_0^{i+1}$ and $x_0^{j+1} x_0^j$ are glued into one cut, the stalks of E_U over points of these edges must be glued into stalks of E. The matrices Y_1 and Y_2 are transformed into two collections of horizontal sections over this cut. The initial conditions for these horizontal sections coincide, since the collection of sections $Y_1(x_0^i)$ of the bundle E_U over the point x_0^i is identified with the collection of sections $S_{x_0^i x_0^{j+1}} Y_1(x_0^i) = Y_2(x_0^{j+1})$ over the point x_0^{j+1}. Then the collections of sections Y_1 and Y_2 must be glues together over the whole cut. It follows that if the points $z \in x_0^i x_0^{i+1}$ and $z' \in x_0^{j+1} x_0^j$ are glued under the factorization, then $S_{z,z'} Y_1(z) = Y_2(z')$. The proposition is proved. $\square$

The total space of the bundle E is obtained from the total space E_U in the following way: if the points $z, z' \in \partial U$ are glued under the factorization into the Riemann surface then, we glue the stalks $E_{U,z}$ and $E_{U,z'}$ using the operator $S_{z,z'}$.

The connection ∇ in E is reconstructed automatically. Indeed, $\operatorname{int} U$ is mapped biholomorphicly onto some open dense subspace U' in M. Hence the connection ∇_U uniquely defines a connection ∇ in $E|_{U'}$. Since the set U' is dense and in the set $\partial U'$ there are no singularities, the connection is uniquely defined on the whole surface.

We have proved the following statement.

Proposition 4. *From a form ω of type* (1) *with singularities in the fundamental polygon U and matrices $S_{x_0^1, x_0^i}$, $i = 2, \dots, 4g$, obtained from a pair (E, ∇) on a surface, the pair (E, ∇) is reconstructed as follows:*

1) we construct a pair (E_U, ∇_U) -which is a trivial bundle on U with a connection defined by the form ω;

2) using the rule described in the proposition 3 using the matrices $S_{x_0^1, x_0^i}$ we reconstruct the matrices $S_{z,z'}$ for all pairs of points $z, z' \in \partial U$, that are glued under the factorization;

3) the total space E is obtained from the total space of E_U using the following rule: if $z, z' \in \partial U$ are glued together under the factorization into the Riemann surface, then we glue the stalks $E_{U,z}$ and $E_{U,z'}$ using the operators $S_{z,z'}$;

4) the connection ∇ in E is reconstructed automatically.

Express the monodromy of ∇ along the cuts using the form $\omega$and matrices $S_{x_0^1, x_0^i}$. Denote by (E_U, ∇_U) a trivial bundle on U with a connection defined by ω.

Definition 8. Take in a stalk E_{U,x_0^1} over the point x_0^1 a base $e_1^1, \dots, e_p^1$. Then in the stalk E_{U,x_0^i} over a point x_i^0 we obtain a base $S_{x_0^1,x_0^i} e_1^1, \dots, S_{x_0^1,x_0^i} e_p^1$. Such a system of bases in the stalks E_{U,x_0^i}, $i = 1, \dots, 4g$ we call a coherent system of bases.

When we identify $E_{U,x_0^i} = E_{x^0}$, all these bases are identified with one base in E_{x^0}, we denote it $e_1, \dots, e_p$.

Take as a base in E_{U,x_0^1} a standard base

$$e_1^1 = (1, 0, \dots, 0), \qquad \dots, \qquad e_p^1 = (0, 0, \dots, 1).$$

Take a coherent system of bases in E_{U,x_0^i}, $i = 1,\dots,4g$. This gives as a base $e_1,\dots,e_p$ in E_{x^0}. Find the monodromy matrices in this base.

Lemma 1. *Consider horizontal sections $y_1,\dots,y_p$ starting at the point x_0^1 then going along the curve $x_0^1 x_0^2 \dots x_0^{i-1} x_0^i$ with the initial condition $y_k(x_0^1) = e_k^1$, $k = 1,\dots,p$. Write then in a matrix $Y = (y_1,\dots,y_p)$. Then the monodormy matrix corresponding to the bypass along the curve, obtained from $x_0^1 x_0^2 \dots x_0^{i-1} x_0^i$ after the factorization, equals $S^{-1}_{x_0^1,x_0^i} Y(x_0^i)$.*

Proof. Indeed, $S^{-1}_{x_0^1,x_0^i} Y(x_0^i)$ is a matrix in a base $y_k(x_0^1) = e_k^1$, $k = 1,\dots,p$, of the operator that firstly does the horizontal transportation of sections along the curve from the stalk over $x_0^1$to the stalk over x_0^i, and then identifies the stalks as under the factorization into E. By definition this matrix is the monodromy matrix in the base $e_1,\dots,e_p$ along the curve that we obtain under the factorization $x_0^1 x_0^2 \dots x_0^{i-1} x_0^i$. The lemma is proved. □

Below, when we say "the monodromy of the bypass along the loop $x_0^1 x_0^2 \dots x_0^{i-1} x_0^i$", we shall mean the monodromy of the bypass along the loop that we obtain from $x_0^1 x_0^2 \dots x_0^{i-1} x_0^i$ under the factorization.

If we know monodromies of the bypasses along all loops $x_0^1 x_0^2 \dots x_0^{i-1} x_0^i$, then we know the monodromy of the bypass along each loop $x_0^j x_0^{j+1}$. Thus we shall write the condition that the monodromy is preserved we shall write the condition that the monodromy of the bypasses along all loops $x_0^1 x_0^2 \dots x_0^{i-1} x_0^i$ is preserved. Below we need also an expression for the monodromy of the bypass along the loop $x_0^i x_0^{i+1}$.

Lemma 2. *The monodromy matrix of ∇ of the bypass along the loop that we get from $x_0^i x_0^{i+1}$ after the factorization is written as follows: take horizontal sections $\tilde{y}_1,\dots,\tilde{y}_p$ of the pair (E_U,∇_U) starting from x_0^i going along $x_0^i x_0^{i+1}$ such that $\tilde{y}_k(x_0^i) = e_k^i$, $k = 1,\dots,p$, write them in a matrix $\widetilde{Y} = (\tilde{y}_1,\dots,\tilde{y}_p)$ (note that $\widetilde{Y}(x_0^i) = S_{x_0^1 x_0^i}$). Then the monodromy along $x_0^i x_0^{i+1}$ equals $\widetilde{Y}^{-1}(x_0^i) S^{-1}_{x_0^i,x_0^{i+1}} \widetilde{Y}(x_0^{i+1}) = S^{-1}_{x_0^1 x_0^{i+1}} \widetilde{Y}(x_0^{i+1})$.*

is analogous to the proof of the previous lemma. □

Up to now we have suggested that the form ω and matrices $S_{x_0^1,x_0^i}$ are obtained from a pair (E,∇) on a Riemann surface. Let us now give an answer to the following natural question. Let us be given a Riemann surface presented as a factor of the exterior of the unit disc. Let us be given a form (1) on the Riemann sphere, such that all its singularities, except may be zero, belong to the fundamental polygon U, let us be given a collection of nondegenerate matrices $S_{x_0^1,x_0^i}$, $i = 1,\dots,4g$. In the Proposition 4 we have described a procedure how to reconstruct a bundle with a connection on the Riemann surface from this data. The question is for which data this procedure is correct?

Lemma 3. *A necessary and sufficient condition for the possibility of reconstruction of (E,∇) using the procedure from the proposition 4 is the following:*

let $x_0^i x_0^{i+1}$ and $x_0^{j+1} x_0^j$ be glued into one canonical cut on the Riemann surface. Then

$$Y_1^{-1}(x_0^i) S^{-1}_{x_0^i, x_0^{i+1}} Y_1(x_0^{i+1}) = Y_2^{-1}(x_0^{j+1}) S^{-1}_{x_0^{j+1},\, x_0^j} Y_2(x_0^j), \tag{3}$$

where Y_1 is a matrix whose columns are horizontal sections of the pair (E_U, ∇_U) along $x_0^i x_0^{i+1}$ with the initial condition $Y_1(x_0^i) = S_{x_0^1 x_0^i}$, and Y_2 is a matrix whose columns are horizontal sections of the pair (E_U, ∇_U) along $x_0^{j+1} x_0^j$ with the initial condition $Y_2(x_0^{j+1}) = S_{x_0^1 x_0^{j+1}}$.

Proof. Let us prove that the condition is necessary. Since $x_0^i x_0^{i+1}$ and $x_0^{j+1} x_0^j$ are glued into one cut, the monodromy of the bypass along this cut can be calculated using $x_0^i x_0^{i+1}$ or $x_0^{j+1} x_0^j$, but the result must be the same. The monodromy calculated using $x_0^i x_0^{i+1}$ equals $Y_1^{-1}(x_0^i) S^{-1}_{x_0^i, x_0^{i+1}} Y_1(x_0^{i+1})$, the monodromy calculated using $x_0^{j+1} x_0^j$, equals $Y_2^{-1}(x_0^{j+1}) S^{-1}_{x_0^{j+1}, x_0^j} Y_2(x_0^j)$. The condition (3) just says that these two expressions are equal.

Now let us prove that the condition is sufficient. We can always construct a pair (E_U, ∇_U) which is a trivial bundle with a connection defined by the form ω. Since $\operatorname{int} U$ is mapped biholomorphicly onto some open dense subset $U' \subset M$, we obtain a bundle with a connection on U'. We need to glue it into a bundle with a connection on the whole surface.

Take a point $P = x_0$ and consider its small neighborhood O. At the point P all cuts meet. In the point P the gluing procedure is already defined $S_{x_0^1, x_0^i}$. Using the proposition 2, we can glue along the horizontal section along cuts that are contained in O. As a result we obtain a bundle with a connection (E, ∇) over this small neighborhood O, we need to glue along the rest parts of the cuts that are not contained in O. Again we shall glue along the horizontal sections. The rest part of a cut is a curve, whose ends P_1 and P_2 belong to ∂O. Take one of its ends P_1 as an initial point and do the gluing along the horizontal sections. We need to check that this procedure is correct in the end point P_2.

Now we are in the situation described in the Proposition 2: we take as γ the whole cut (with its part that belongs to O) as V a small neighborhood of the cut. We have already a bundle with a connection in V outside γ and a gluing in $\gamma \cap O$. The correctedness of gluing along γ is equivalent to the coincidence of monodromies of glued connections along γ. But the condition (3) just expresses this coincidence. The lemma is proved. □

In this section we have proved.

Theorem 1. *For the bundle with a connection (E, ∇) on a Riemann surface we have constructed the following data:*

1) a form (1) on sphere, all singularities of the form, except may be zero, belong to the fundamental polygon U, and zero is a regular singularity;

2) a collection of matrices $S_{x_0^1, x_0^i}$, wherex_0^i are vertices of the fundamental polygon, $i = 1, \ldots, 4g$, and $S_{x_0^i, x_0^{i+1}} = S_{x_0^1 x_0^{i+1}} S^{-1}_{x_0^1 x_0^i}$ satisfy the condition (3) for all pairs of edges $x_0^i x_0^{i+1}$ and $x_0^{j+1} x_0^j$, that are glued into one cut. The matrices Y_1, Y_2 are the same as in Lemma 3.

The inverse is true: using such data one can construct a bundle with a connection on a surface.

4 Isomonodomic deformation

In the previous sections pair a (E, ∇) on the surface we have constructed a form (1) on a Riemann sphere with a regular singularity in zero and matrices $S_{x_0^1, x_0^i}$. Now we show how the isomonodromic deformations of pairs (E, ∇) are described using such correspondence.

Let (E^1, ∇^1) -be a pair on $\widetilde{T}_1$. For $\tau \in \widetilde{T}$ denote by $(E^1, \nabla^1)|_\tau$ a restriction of the bundle with a connection (E^1, ∇^1) to the subspace $\widetilde{T}_1|_\tau$ (see Definition 3).

Definition 9. An isomonodormic family is pair (E^1, ∇^1) on $\widetilde{T}_1$ such that:

1) a pair (E^1, ∇^1)has singularities on hypersurfaces $z = a_i$ (more precise, on hypersurfaces in $\widetilde{T}_1$, that are preimages of hypersurfaces $z = a_i$ in T_1);

2) for all $\tau \in \widetilde{T}$ pairs $(E^1, \nabla^1)|_\tau$ have the same monodormy.

Let us be given a point $\tau_0 \in \widetilde{T}$. Let (E, ∇) be a pair on a marked Riemann surface, corresponding to τ_0, the singularities of ∇ correspond to marked points of τ_0.

Definition 10. We say that an isomonodromic family (E^1, ∇^1) describes a deformation of a pair (E, ∇), if $(E^1, \nabla^1)|_{\tau_0} = (E, \nabla)$.

Definition 11. A family (E^1, ∇^1) is called Schlesinger, if for a fixed point in the Teichmuller space in some neighborhood of the hypersurface $z = a_i$ a connection ∇^1 is written in local coordinates as a form of type

$$\frac{B_i}{\zeta - a_i}\, d(\zeta - a_i) + h(\zeta, a_i),$$

where $h(\zeta, a_i)$ is a holomorphic form, B_i are holomorphic functions of a_i.

Let us state a result about global existence of deformations.

Proposition 5. *For every logarithmic initial pair (E, ∇) at $t_0 \in \widetilde{T}$ there exists a unique its continuation to the Schlesinger isomonodromic family (E^1, ∇^1).*

Proof. Since this proposition is well-known let us give only a sketch of a proof. We must construct a pair (E^1, ∇^1) on $\widetilde{T}_1$. First of all note that there is an isomorphism $\pi_1(M \setminus \{a_1^0, \ldots, a_n^0\}) \to \pi_1(\widetilde{T}_1)$ (here $a_1^0, \ldots, a_n^0$ are initial positions of singularities); this is a corollary of the homotopic equivalence . Using the Rörl construction [13], one cam construct a pair (E^1, ∇^1) on $\widetilde{T}_1$ outside small neighborhoods of hypersurfaces $z = a_i$. The problem of a construction of (E^1, ∇^1)in a neighborhood of a hypersurface $z = a_i$ is local, we can use the analogous construction in the case of the Riemann sphere. Thus we obtain a pair (E^1, ∇^1) on $\widetilde{T}_1$. □

Establish a relation between a pair (E^1, ∇^1) on $\widetilde{T}_1$ and a family of forms from the theorem 1. Consider the Schlelinger family (E^1, ∇^1) and a point $t^0 \in \widetilde{T}_1$. Denote as τ^0 singularities, corresponding to t^0 as $a_1^0, \dots, a_n^0$, and a point in the Teichmullr space, corresponding to t^0. Let $W_{a_i^0}$ be a sufficiently small neighborhood of the point a_i^0 (such that $W_{a_i^0} \cap W_{a_j^0} = \varnothing$, if $i \neq j$), $i = 1, \dots, n$, and V_{τ^0} be a sufficiently small neighborhood of the point τ^0 in the Teichmuller space.

Proposition 6. *There exists a form ω^1 on the space*

$$\bigl\{(z, a_1, \dots, a_n, \tau) \colon z \in \overline{\mathbb{C}},\ a_i \in W_{a_i^0},\ i = 1, \dots, n,\ t \in V_{\tau^0}\bigr\},$$

with the following properties:

1) the following presentation takes place

$$\omega^1 = \frac{C_k}{z^k}\, dz + \dots + \frac{C_1}{z}\, dz + \sum_{i=1}^{n} \frac{B_i}{z - a_i}\, d(z - a_i); \tag{4}$$

2) if we fix a point t, which is sufficient close to t^0, and consider a pair $(E^1, \nabla^1)|_t$ and a form ω, corresponding to the pair $(E^1, \nabla^1)|_t$ by theorem 1, then the form ω can be obtained by fixing singularities in the form ω^1 in singularities, corresponding to the point t[1]*.*

Proof. Take an intersection of all fundamental polygons that correspond to all points in the Teichmuller space, that belong to V_τ. Let O be an open neighborhood of this intersection, whose boundary is smooth simple curve. Suppose that the neighborhood V_{τ^0} is so small that $W_{a_i^0} \subset O$ for all $i = 1, \dots, n$.

There exist a mapping

$$f \colon O \times W_{a_1^0} \times \dots \times W_{a_n^0} \times V_\tau \to T_1, \tag{5}$$

which is defined in the following way. Take a point

$$(z, a_1, \dots, a_n, t) \in O \times W_{a_1^0} \times \dots \times W_{a_n^0} \times V_\tau.$$

Using a point t in the Teichmuller space, one can reconstruct a pair $G_t \subset \operatorname{Aut} D$ (remind that D is the exterior of the unit disc). Denote as z/G_t the image of $z \in O \subset D$ on the marked Riemann surface under the factorization $D \to M$ under the action of G_t. Then f maps the point $(z, a_1, \dots, a_n, t)$ to the point $(t, z/G_t, a_1/G_t, \dots, a_n/G_t) \in T_1$. Since the image of this mapping is small enough, the mapping (5) is well defined. Note that this mapping is not holomorphic, but it becomes holomorphic if we fix a point in the Teichmuller space.

Take in inverse image of (E^1_O, ∇^1_O) of the pair (E^1, ∇^1) under the mapping f. If a point in the Teichmuller space is fixed, this is a holomorphically trivial bundle with a connection.

[1] It is important to note that the form ω^1 does not contain coordinates on V_{τ^0} and differentials of these coordinates.

When the parameters $a_1, \ldots, a_n, \tau$ are fixed one can continue (E^1_O, ∇^1_O) from the subset O to the whole Riemann sphere $\overline{\mathbb{C}}$. Note that the connection ∇^1_O depends holomorphically from $a_1, \ldots, a_n$, and the bundle E^1_O does not depend on $a_1, \ldots, a_n$. Hence for fixed $\tau \in V_{\tau^0}$ the holomorphic pair (E^1_O, ∇^1_O) can be continued from $O \times W_{a_1^0} \times \cdots \times W_{a_n^0} \times \{\tau\}$ to a holomorphic pair on $\overline{\mathbb{C}} \times W_{a_1^0} \times \cdots \times W_{a_n^0} \times \{\tau\}$.

By the Grotendick-Birkhoff theorem with parameters [1] there exists a meromorphic trivialization of this bundle, which is holomorphic on $\overline{\mathbb{C}} \setminus \{0\} \times W_{a_1^0} \times \cdots \times W_{a_n^0} \times \{\tau\}$. The form ω^1 is the form of the connection ∇^1_Oin this trivialization. Note that ω^1 can (in a non-holomorphic way) depend on τ. Remind that (E^1, ∇^1) is a Schlesinger family. Hence for a fixed τ the form ω^1 is of type

$$\frac{C_k}{z^k}\, dz + \cdots + \frac{C_1}{z}\, dz + \sum_{i=1}^{n} \frac{B_i}{z - a_i}\, d(z - a_i) + \sum_{i=1}^{n} D_i\, da_i.$$

Choose another trivialization in which coefficients at da_i vanish. It is constructed in the following way. The family (E^1, ∇^1) is isomonodromic, hence the family of forms ω^1 is also isomonodromic. This is equivalent of the fact that $d\omega^1 = \omega^1 \wedge \omega^1$, i.e. the form ω^1 is integrable. Let $(p \times p)$ matrix $Y_0(z, a_1, \ldots, a_n)$ be a solution of $dY_0 = \omega^1 Y_0$ (the differential is taken by the variables $a_1, \ldots, a_n$) with the initial condition $Y_0(\infty, a_1^0, \ldots, a_n^0) = I$. Let $Y_0^\infty = Y_0(\infty, a_1, \ldots, a_n)$. Define a new trivialization of E^1_O on $\overline{\mathbb{C}} \setminus \{0\} \times W_{a_1^0} \times \cdots \times W_{a_n^0} \times \{\tau\}$, by acting on the old base by the matrix $(Y_0^\infty)^{-1}$, to obtain a new base in every stalk. The form of the connection ∇^1_O in this new trivialization is the new form ω^1.

By construction in this new trivialization there exist a solution of the system $dY = \omega^1 Y$ such that $Y(\infty, a_1, \ldots, a_n) \equiv I$ for all $a_1, \ldots, a_n$. Since

$$\left.\frac{\partial Y}{\partial a_i}\right|_{z=\infty} = \left.\frac{B_i}{z - a_i}\right|_{z=\infty} + D_i = D_i,$$

we have $D_i = 0$. Thus in new trivialization of E^1_O on the space $\overline{\mathbb{C}} \setminus \{0\} \times W_{a_1^0} \times \cdots \times W_{a_n^0} \times \{\tau\}$ the form ω^1 is written as (4).

Note that the change of the point in the Teichmuller space leads only to the change of the fundamental polygon. But the trivialization of E^1_O in section 3.1 does not depend on its precise shape. Thus the trivialization of E^1_O can be chosen such that the equality (4) for ω^1 takes place on the whole space $\overline{\mathbb{C}} \setminus \{0\} \times W_{a_1^0} \times \cdots \times W_{a_n^0} \times V_\tau$. The Proposition is proved. $\square$

Now let us write the equations of isomonodromic deformations. The first group of equations describes the Schlesinger deformations of the form (1). Obviously when the pair (E, ∇) deforms isomonodromically, the form ω also deforms isomonodromically.

Consider the Schlesinger deformations. Then the deformations of ω are defined by the form ω^1 of type (4). The form ω^1 defines an isomonodromic deformation if and only if $d\omega^1 = \omega^1 \wedge \omega^1$. Note that ω^1 does not contain coordinates on the Tecihmuller space and differentials of these coordinates, hence the differential is taken only by variables z, a_i.

Proposition 7. *The equations of isomonodromic deformations of ω are*

$$dB_i = -\sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{[B_i, B_j]}{a_i - a_j}\, d(a_i - a_j) + \frac{\partial C_1}{\partial a_i}\, da_i, \tag{6}$$

$$\frac{\partial C_2}{\partial a_i} - \frac{\partial C_1}{\partial a_i} a_i = -[B_i, C_1], \qquad \dots, \qquad \frac{\partial C_{l+1}}{\partial a_i} - \frac{\partial C_l}{\partial a_i} a_i = -[B_i, C_l], \tag{7}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$-\frac{\partial C_k}{\partial a_i} a_i = -[B_i, C_k], \qquad i = 1, \dots, n. \tag{8}$$

In the typical case $C_l = 0$, $l > 1$, if we put $a_0 = 0$, $B_0 = C_1 = -\sum_{i=1}^{n} B_i$, the system above turns into an ordinary Schlesinger system

$$B_i = -\sum_{\substack{j=0,\\ j\neq i}}^{n} \frac{[B_i, B_j]}{a_i - a_j}\, d(a_i - a_j). \tag{9}$$

Here $i = 0, 1, \dots, n$.

Proof. Explicitly the equality $d\omega^t = \omega^t \wedge \omega^t$ can be written as follows. At first let us write the left hand side:

$$\sum_{i=1}^{n} \frac{dB_i}{z - a_i}\, d(z - a_i) + \sum_{l=1}^{k} \frac{dC_l}{z^l}\, dz = \sum_{i,j=1}^{n} \frac{\partial B_i}{\partial a_j} \frac{1}{z - a_i}\, dz \wedge d(z - a_i) -$$

$$- \sum_{i,j=1}^{n} \frac{\partial B_i}{\partial a_j} \frac{1}{z - a_i}\, d(z - a_j) \wedge d(z - a_i) + \sum_{l=1}^{k} \sum_{i=1}^{n} \frac{\partial C_l}{\partial a_i} \frac{1}{z^l}\, dz \wedge d(z - a_i).$$

Now let us write the right hand side:

$$\left(\sum_{i=1}^{n} \frac{B_i}{z - a_i}\, d(z - a_i) + \sum_{l=1}^{k} \frac{C_l}{z^l}\, dz \right) \wedge \left(\sum_{j=1}^{n} \frac{B_j}{z - a_j}\, d(z - a_j) + \sum_{l=1}^{k} \frac{C_l}{z^l}\, dz \right) =$$

$$= \sum_{\substack{i,j=1,\\ i\neq j}}^{n} \frac{B_i B_j}{(a_i - a_j)} \left(\frac{1}{z - a_i} - \frac{1}{z - a_j} \right) d(z - a_i) \wedge d(z - a_j) +$$

$$+ \sum_{l=1}^{k} \sum_{i=1}^{n} \frac{[B_i, C_l]}{(z - a_i) z^l}\, d(z - a_i) \wedge dz.$$

The forms $d(z - a_i) \wedge d(z - a_j)$ Ë $dz \wedge d(z - a_i)$ are independent [1], so the coefficients at these forms on the right and on the left must coincide. The coincidence of the coefficients at the form $dz \wedge d(z - a_i)$ gives the equation

$$\sum_{j=1}^{n} \frac{\partial B_i}{\partial a_j} \frac{1}{z - a_i} + \sum_{l=1}^{k} \frac{\partial C_l}{\partial a_i} \frac{1}{z^l} = -\sum_{l=1}^{k} \frac{1}{z^l} \frac{[B_i, C_l]}{z - a_i}.$$

Multiply it by $(z - a_i)$, one gets

$$\sum_{j=1}^{n} \frac{\partial B_i}{\partial a_j} + \sum_{l=1}^{k} \left(\frac{1}{z^{l-1}} - a_i \frac{1}{z^l} \right) \frac{\partial C_l}{\partial a_i} = -\sum_{l=1}^{k} \frac{1}{z^l} [B_i, C_l].$$

Consider the coefficient at powers of $1/z$: for z^0 we obtain the equation

$$\sum_{j=1}^{n} \frac{\partial B_i}{\partial a_j} + \frac{\partial C_1}{\partial a_i} = 0, \tag{10}$$

for $1/z, \dots, 1/z^l$ we obtain equations (7) and at last for $1/z^k$ we obtain the equation (8).

The coincidence of coefficients at forms $d(z-a_j) \wedge d(z-a_i)$ gives equations that does not contain C_k, they are equivalent to the following equations

$$\frac{\partial B_i}{\partial a_j} = \frac{[B_i, B_j]}{a_i - a_j}, \qquad i \neq j.$$

These equations together with (10) can be written together as the equation (6).

In the case $C_2 = \cdots = C_k = 0$ the derivation above just reproduce the derivation of the Schlesinger equations for the deformations of the form (2) (see [1]). In particular in notations $a_0 = 0$, $B_0 = C_1 = -\sum_{i=1}^{n} B_i$ we obtain an ordinary Schlesinger system (9). The proposition is proved. $\square$

The equations (6)–(8) are uniquely solvable for every initial conditions just because they describe the Schlesinger deformations of the connection $\nabla_{\overline{\mathbb{C}}}$ on a Riemann sphere.

Now let us write the second group of equation that describe the evolution of matrices $S_{x_0^1, x_0^i}$. In lemma 1 it was proved that $S^{-1}_{x_0^1, x_0^i} Y(x_0^i)$ is the monodromy matrix along the loop, that we obtain after the factorization $x_0^1 x_0^2 \dots x_0^{i-1} x_0^i$ (here Y is the same as in Lemma 1).

Take an isomonodromic family of solutions $dY = \omega^1 Y$ such that in the initial position of singularities a_j in the point $z = \infty$ the matrix Y is identical (remind that in this equation the differential is taken only by variables z and a_j, $j = 1, \dots, n$). The form ω^1, defined in (4), is such that the matrix Y in $z = \infty$ is constantly identical. This follows from the fact that $\frac{\partial Y}{\partial a_i}\big|_{z=\infty} = \frac{B_i}{z-a_i}\big|_{z=\infty} = 0$. That is why we can assume that the matrix Y, through which the monodromy in Lemma 1 is expressed, satisfies $dY = \omega^1 Y$.

The monodromy along each cut must be conserved. Hence $S^{-1}_{x_0^1, x_0^i} Y(x_0^i) =$ const or equivivalently, $S_{x_0^1, x_0^i} = \mathrm{const}^{-1} \cdot Y(x_0^i)$. Hence the matrix $S_{x_0^1, x_0^i}$ satisfies the same equation, as $Y(x_0^i)$, but with another initial condition. In other words $d_{x_0^i, a_j} S_{x_0^i x_0^{i+1}} = \omega^1|_{z \mapsto x_0^i} S_{x_0^i x_0^{i+1}}$. Here $\omega^1|_{z \mapsto x_0^i}$ is the form ω^1, in which the variable z is replaced to x_0^i. Namely

$$\omega^1|_{z \mapsto x_0^i} = \sum_{j=1}^{n} \frac{B_j}{x_0^i - a_j} d(x_0^i - a_j) + \left(\frac{C_k}{{x_0^i}^k} + \cdots + \frac{C_0}{x_0^i} \right) dx_0^i.$$

Now let us state the main result of the present paper.

Theorem 2. *For every initial logarithmic pair (E, ∇) on a marked Riemann surface there exists a unique Schlesinger isomonodromic family (E^1, ∇^1). In*

terms of data from the Theorem 1 The Schlesinger isomonodromic deformations are locally described as follows.

1. The vertices of the fundamental polygon change their posiitons according to the change of a point in the Teichmuller space.

2. The evolution of the coefficients C_l, B_i of the form (1) *is given by the equations from the proposition 7. In the case $C_l = 0$, $l > 0$, in notations $a_0 = 0$, $B_0 = -\sum_{i=1}^{n} B_i$ these equations are the Schlesinger system for B_i, $i = 0, \dots, n$.*

3. The evolution of matrices $S_{x_0^1,x_0^i}$ is described by equations $dS_{x_0^1,x_0^i} = \omega^1 S_{x_0^1,x_0^i}$, where ω_1 is the form (4)*, in which instead of z we write the variable x_0^i. The differential in the left side is taken by the variables a_j, x_0^i.*

One ca easily see that the bundle in the pair (E, ∇) changes, thus the case of higher genus differs much from the case of genus 0, where it is natural to consider connetions in a fixed trivial bundle.

5 Relation to the Krichiver's approach

Let us give a short comparision of the approach to the description of isomonodromic deformations of bundles with connection suggested in the present paper with the approach suggested by Kricheve [5]. More precise we are going to show how using the parameters that Krichever has used one can reconstruct the form and matrices from Theorem 1.

Restrict ourself to the case when the marked Riemann surface is fixed. The reason is that in [5] the complex-analytic coordinates on the Teichmuller space are used, and in the present paper we take real-analytic coordinates on the Teichmuller spaces x_0^i which are positions of vertices of the fundamental polygon. The matrices$S_{x_0^1,x_0^i}$ are complex analytic functions of the variables x_0^i, $i = 2, \dots, 4g$. Thus in order to establish a relation between the Krichiver's approach and our approach in general case one must be able to express explicitly positions of vertices of the fundamental polygon through the complex-analytic coordinates on the Teichmuller space. This problem for $g > 1$ is extremely difficult (see for example a close to this problem paper [14]).

Thus let a marked Riemann surface be fixed. Let us be given a stable bundle with a logarithmic connection (E, ∇), where E has rank $p$and degree pg. In the paper [5] for the bundle with a connections some parameters are constructed, these parameters can be divided into two groups. The parameters from the first group describe the bundle (the Turin parameters), and parameters of the second group describe a connection in it (see § 2 in [5]).

The Turin parameters are defined as follows. There exists a meromorphic trivialization of a stable bundle of degree g. This is a collection of holomorphic sections $\psi_1, \dots, \psi_p$. In stalks of E over all points except $\gamma_1, \dots, \gamma_m$ (where $m = pg$), they form a base in the stalk of E. In the stalks over $\gamma_1, \dots, \gamma_m$ these sections are linearly dependent. In the typical case the rank of the span of sections $\psi_1, \dots, \psi_p$ in these points equals $p - 1$, that is there exists a unique linear relation $a_1^i \psi_1(\gamma_i) + \dots + a_p^i \psi_p(\gamma_i) = 0$. The Turin parameters are: the

collection of points $\gamma_1, \dots, \gamma_m$ and the vectors $(a_1^i, \dots, a_p^i)$, $i = 1, \dots, m$. In the meromorphic trivialization $\psi_1, \dots, \psi_p$ to the connection ∇ there corresponds the form $\widetilde{\omega}$ with apparent singularities in points $\gamma_1, \dots, \gamma_m$and a trivial monodormy of the bypass around them. In the paper [5] it is suggested that the typical case takes place and these apparent singularities of $\widetilde{\omega}$ are simple poles.

The parameters of the second group describe the connection ∇. Among these parameters there are those that describe the behavior of $\widetilde{\omega}$ in a neighborhood of γ_i (see. § 2 and the Lemma 2.2 in [5]). In particular using these parameters and the Turin parameters one can reconstruct a residue of the form $\widetilde{\omega}$ in γ_i. Also there are parameters that define behavior of $\widetilde{\omega}$ in a neighborhood of singularities of the connections (see § 4 in [5]). Among these parameters there are the position of singularities and singular parts of $\widetilde{\omega}$ in these neighborhoods.

In the present paper for a pair (E, ∇) we have constructed a form ω and matrices $S_{x_0^1, x_0^i}$. Note that in contrast with Krichever's parameters one can not say that the form ω defines a connection and matrices $S_{x_0^1, x_0^i}$ define a bundle, since in the procedure of reconstruction of the bundle the from ω participates.

Now let us establish a relation between descriptions of (E, ∇), suggested in the present paper and the description suggested by Krichever. At first we construct a form and matrices to the meromorphically trivialized bundle. Using the trivialization $E|_U$, defined by sections $\psi_1, \dots, \psi_p$, we obtain that the from is an inverse image of $\widetilde{\omega}$ on the fundamental polygon (we denote it also as $\widetilde{\omega}$) and, since the bundle is trivial, identity matrices $\widetilde{S_{x_0^1, x_0^i}} = I$.

Note that the residues of the form $\widetilde{\omega}$ in all singularities are contained in Krichevers parameters, thus the form $\widetilde{\omega}$ on the fundamental polygon can be explicitly reconstructed from Krichevers parameters. Now describe how we can reconstruct the form ω and matrices $S_{x_0^1, x_0^i}$ from $\widetilde{\omega}$ and $\widetilde{S_{x_0^1, x_0^i}} = I$. To reconstruct ω, we must apply to $\widetilde{\omega}$ a gauge transformation Γ, which must remove apparent singularities of $\widetilde{\omega}$ in points of the fundamental polygon, corresponding to the points $\gamma_1, \dots, \gamma_m$ of the surface. The transformation Γ can have singularities only in these points and $\Gamma(x_0^1) = I$. Such transformation Γ can be found using the residues of $\widetilde{\omega}$.

To reconstruct $S_{x_0^1, x_0^i}$, we need to act on $\widetilde{S_{x_0^1, x_0^i}} = I$ by the same gauge transformation by the rule $\widetilde{S_{x_0^1, x_0^i}} = I \mapsto S_{x_0^1, x_0^i} = \Gamma(x_0^i)$. Indeed, $\widetilde{S_{x_0^1, x_0^i}}$ is a matrix of the operator that identifies the stalks, hence after the base change according to the definition 6 we obtain the matrix $S_{x_0^1, x_0^i} = \Gamma(x_0^i)\widetilde{S_{x_0^1, x_0^i}}\Gamma(x_0^1)^{-1}$. But two matrices in the right side are identity matrices, hence we obtain the requied expression.

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